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BEZIER CURVES ON GROUPS

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Abstract—A sequence of views, or key frames, of a scene, may be described by transformations $\{T_1, T_2, \ldots, T_n\}$. The T_i will be elements of some continuous group G, for example the rotation or Euclidean or projective linear group. We wish to smoothly animate this sequence of key frames, that is, construct a smooth curve, lying in G, which passes through each T_i. The classical Bezier spline construction is extended from $\mathbb{R}^n$ to the continuous group G using Lie algebra techniques.

1. INTRODUCTION

In [1], Shoemake has shown how key rotations $R_1, \ldots, R_n$ may smoothly interpolated by a one-parameter family of rotations. The technique used involves treating rotations as unit quaternions, which are points on the unit 3-sphere. Bezier interpolation can be generalized from the Euclidean to the spherical case by replacing straight line segments by arcs of great circles. A class of spherical Bezier curves is constructed, which, in the case of a commutative subgroup of rotations, corresponds to the image of ordinary Bezier curves in $\mathbb{R}^3$ under the exponential map.

In applications to computer graphics, one is interested in more general transformations than rotations, for example translations and projections. We are thus led to the following problem: Given a continuous group of transformations G, and a sequence $\{T_1, \ldots, T_n\}$ of elements of G, construct a smooth curve, lying in G and passing through each T_i. Our solution to this problem extends the classical Bezier construction to matrix groups.

2. BEZIER'S CONSTRUCTION

We begin by reviewing Bezier's construction. Given a sequence of control vertices $P_0, \ldots, P_m$ in $\mathbb{R}^3$, we construct a Bezier curve lying in their convex hull and passing through the points P_0 and P_m by the following geometric scheme [2].

The curve to be constructed will be parametrized by

$$s: 0 \leqslant s \leqslant 1.$$

On the $(\mu + 1)$th side $(\mu = 0, \ldots, m-1)$ of the polygon $P_0, \ldots, P_m$, move distance proportional to s from P_μ to $P_{\mu+1}$. Call this set of points p_μ^1 and consider next the polygon $p_0^1, \ldots, P_{m-1}^1$. Repeat this procedure, obtaining a new set of points $P_0^2, \ldots, P_{m-2}^2$. After m steps, a single point P_0^m is obtained, which is the point on the curve corresponding to the parameter value s. This construction is illustrated in Fig. 1.

Algebraically, this construction is given by the recurrence relations

$$P_\mu^m(s) = P_\mu^{m-1} + s[P_{\mu+1}^{m-1} - P_\mu^{m-1}]$$

and hence in terms of the original vertices, the point P_0^m is given by

$$\begin{aligned} p_0^m = {} & (1-s)^m P_0 + s(s-1)^{m-1}\begin{bmatrix} m \\ 1 \end{bmatrix} P_1 \\ & + s^2(1-s)^{m-2}\begin{bmatrix} m \\ 2 \end{bmatrix} P_2 \\ & + \cdots + s^{m-1}(1-s)\begin{bmatrix} m \\ m-1 \end{bmatrix} P_{m-1} + s^m P_m, \end{aligned}$$

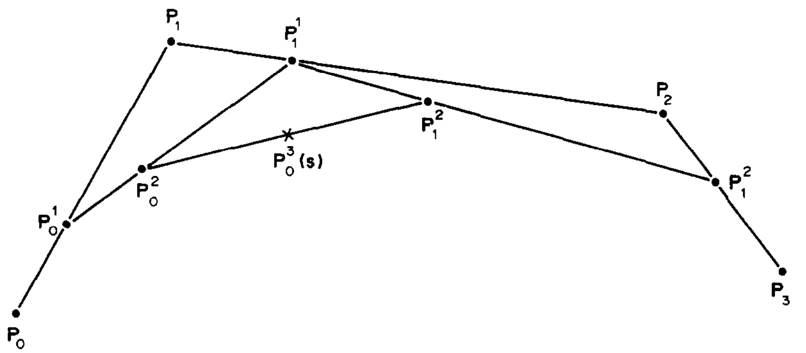

Fig. 1

where $\begin{bmatrix} m \\ n \end{bmatrix}$ is the binomial coefficient

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{m!}{n!(n-m)!}.$$

So

$$P_0^m(s) = \sum \begin{bmatrix} m \\ \mu \end{bmatrix} s^\mu (1-s)^{m-\mu} P_\mu,$$

where $\begin{bmatrix} m \\ \mu \end{bmatrix} s^\mu (1-s)^{m-\mu}$, $\mu = 1, \ldots,$ are the Bernstein polynomials of order m.

Note that for four points P_0, P_1, P_2, P_3, the Bezier curve has the following properties (see Fig. 2):

(1) The curve passes through P_0 and P_3.

(2) The curve is tangent to $\overline{P_0P_1}$ at P_0.

(3) The curve is tangent to $\overline{P_2P_3}$ at P_3.

The above properties allow the construction of Bezier splines which smoothly interpolate a given set of points.

Given points $Q_0, \ldots, Q_n$ we add control vertices b_k, a_k before and after each Q_k, and construct the Bezier curve corresponding to each quadruple $Q_k, a_k, b_{k+1}, Q_{k+1}$ (see Fig. 3). To obtain a_k take the midpoint of the segment between Q_{k+1} and the reflection of Q_{k-1} through Q_k along Q_{k-1}. Thus a_k is the average of where a point would end up going straight from Q_{k-1} through Q_k and Q_{k+1}. Then b_k is the reflection through Q_k of a_k (see Fig. 4). Piecing these segments of Bezier curves together gives a Bezier spline. Since b_k, Q_k, a_k are collinear, the joint at each Q_k will be smooth. The resulting curve passes smoothly through each $Q_0, \ldots, Q_n$.

In order to extend this construction to matrix groups, we need to find the analogues of parametrized straight line segments and reflections. These will be obtained using the exponential map.

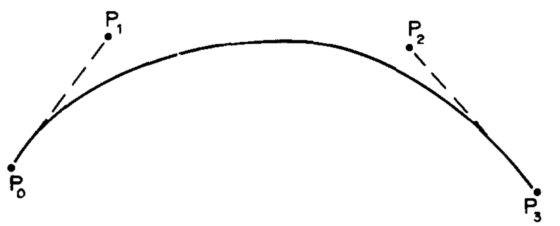

Fig. 2

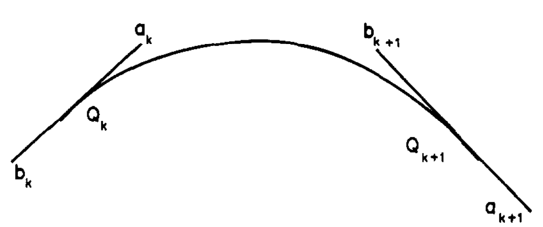

Fig. 3

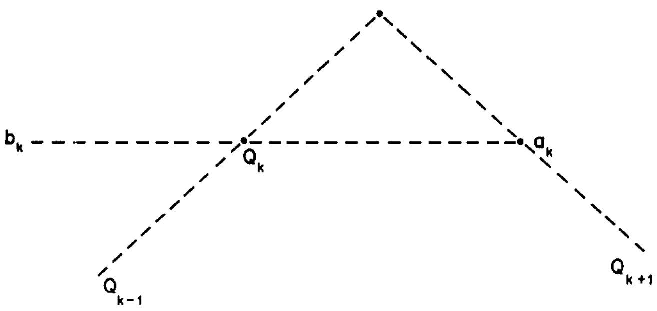

Fig. 4

3. BASIC PROPERTIES OF THE EXPONENTIAL MAP

In this section we sketch the basic properties of the exponential map that we will subsequently need [3]. G will denote a continuous matrix group [4]. For example, the group

$$\mathrm{SO}(3) = \{A \quad 3 \times 3 \text{ matrix} \mid AA^t = I\} \quad \text{or}$$
$$\mathrm{SL}_4 = \{A \quad 4 \times 4 \text{ matrix} \mid \det A = 1\},$$

which represent rotations (respectively projective linear transformations) [5]. Bezier's constructions involve successive linear interpolations, but a continuous group is a curved maniford and we will not in general have straight lines. There is, however, a natural generalization of a parametrized straight line, which is a one-parameter subgroup. Using one-parameter groups we will be able to repeat Bezier's construction.

All groups under consideration will be subgroups of the general linear group $GL_n = \{A \ n \times n$ matrix $\mid \det A \neq 0\}$. For simplicity we restrict our subsequent considerations to $G = GL_n$.

If A is an $n \times n$ matrix the exponential $\exp A$ is defined by the power series

$$\exp A = I + A + A^2/2! + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where I is the $n \times n$ identity matrix and the above series converges absolutely (that is, each matrix entry of the series converges absolutely).

More generally, for any real u,

$$\exp \mathrm{u}A = \sum_{k=0}^{\infty} \frac{\mathrm{u}^k A^k}{k!}$$

converges. Example:

$$\exp\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \text{ (all higher powers)}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix};$$
$$\exp\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \exp(\lambda_1) & 0 \\ 0 & \exp(\lambda_2) \end{bmatrix}$$
$$\exp\begin{bmatrix} 0 & \mathrm{u} \\ -\mathrm{u} & 0 \end{bmatrix} = \begin{bmatrix} \cos \mathrm{u} & \sin \mathrm{u} \\ -\sin \mathrm{u} & \cos \mathrm{u} \end{bmatrix}.$$

Basic properties of the exponential map

Property 1.

$$\det \exp A = \exp(\operatorname{trace} A) \neq 0 \quad \text{for any } A$$

so $\exp A$ is in GL_n for any $n \times n$ matrix.

Property 2.

$$\exp(u_1 + u_2)A = \exp u_1 A \exp u_2 A,$$
$$\exp(-uA) = (\exp uA)^{-1},$$
$$\exp 0A = I,$$

so that for fixed A $\exp uA$ is a homomorphism of $(R, +)$ into GL_n. $\{\exp uA \mid u \in \mathbb{R}\}$ is called a one-parameter subgroup of GL_n.

Property 3.

$$\frac{d}{du} \exp uA = A \exp uA, \tag{3}$$

so

$$\frac{d}{du} \exp uA \,|_{u=0} = A.$$

A is called the infinitesimal generator of the one-parameter subgroup $\{\exp uA\}$, and $\exp uA$ is the (unique) solution to the differential equation

$$\frac{dY}{du} = AY,$$
$$Y(0) = I.$$

Thus, for $u \in [0, 1]$, $u \mapsto \exp uA$ is a smooth curve connecting I and $\exp A$, whose "tangent" at $u = 0$ is A.

Note. Since matrices do not in general commute,

$$\exp(A + B) \neq \exp A \exp B.$$

In fact $\exp(A + B) = \exp A \exp B$ only when the commutator $[A, B] \equiv AB - BA = 0$, that is when A and B commute.

Property 4. $A \mapsto \exp A$ is A 1–1 map of a neighborhood U of 0 onto a neighborhood V of I in GL_n (see Fig. 5). Thus on V the inverse log of exp can be defined, and is given by the power series

$$\log X = (X - I) - \frac{(X - I)^2}{2} + \frac{(X - I)^3}{3} + \cdots$$

Remark. A norm on GL_n is given by $\|A\| = \text{trace } AA^t$, in terms of which distance and neighborhoods may be defined. The logarithm on $GL_n(R)$ is defined on a domain containing the open ball of radius one about I in this norm. For the rotation group $SO(3)$, exp is invertible on the set of antisymmetric matrices with eigenvalues of modulus less than π.

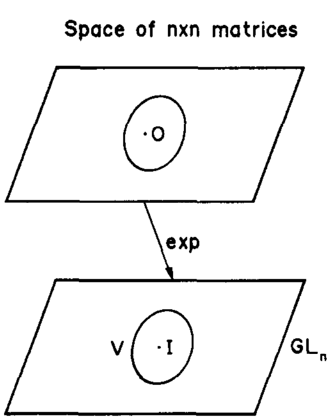

Fig. 5

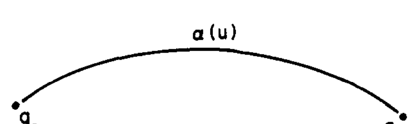

Fig. 6

Property 5: Cambell–Baker–Hausdorff formula.

$$\exp A \exp B = \exp(A + B + C_1 + C_2 + \cdots),$$

where C_k is a linear combination of k-fold commutators of A, B. In particular

$$C_1 = 1/2[A, B],$$

$$C_2 = \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]],$$

so up to quadratic terms

$$\exp A \exp B \approx \exp[A + B + 1/2[A, B]). \tag{1}$$

4. CONSTRUCTION OF BEZIER CURVES ON GL_n

We now apply the results of the previous section to the construction of Bezier curves on GL_n. We show first how the analogue of a parametrized straight line segment joining two points in $\mathbb{R}^n$ can be constructed.

4.1. Two point interpolation

Given g_0, g_1, in GL_n, with $g_0^{-1} g_1$ sufficiently close to I, set

$$A = g_0^{-1} g_1,$$

$$X_0 = \log A,$$

and define the curve in GL_n

$$\alpha(\mathrm{u}) = g_0 \exp \mathrm{u}X_0 \quad 0 \leqslant \mathrm{u} \leqslant 1.$$

Then

$$\alpha(0) = g_0,$$

$$\begin{aligned} \alpha(1) &= g_0 \exp X_0 \\ &= g_0 \exp \log A \\ &= g_0 A \\ &= g_1. \end{aligned}$$

Note

$$\left.\frac{d\alpha}{du}\right| u = 0 = g_0 X_0.$$

The first order Bezier curve $\alpha(\mathrm{u})$ will replace straight line segments in Bezier's construction (Fig. 6.)

4.2. Three point interpolation

Let g_0, g_1, g_2 be elements in GL_n, with g_0, g_1 and g_2 sufficiently close.
Define

$$A = g_0^{-1} g_1 \quad X_0 = \log A = \log g_0^{-1} g_1,$$

$$B = g_1^{-1} g_2 \quad X_1 = \log B = \log g_1^{-1} g_2.$$

Next, for $0 \leqslant \mathrm{u} \leqslant 1$, set

$$g_0^1(\mathrm{u}) = g_0 \exp \mathrm{u}X_0,$$

$$g_1^1(\mathrm{u}) = g_1 \exp \mathrm{u}X_1.$$

Finally, set

$$X_0^1 = \log[(g_0^1(\mathrm{u}))^{-1} g_1^1(\mathrm{u})]$$

and define

$$\alpha(u) = g_0^2(u) = g_0^1(u) \exp uX_0^1,$$
$$= g_0 \exp uX_0 \exp uX_0^1$$

(see Fig. 7).

By repeated application of the Cambell–Hausdorff formula, we will show that $\alpha(u)$ may be approximated in terms of the original fixed values g_0, X_0 and X_1. We have

$$\alpha(u) \approx g_0 \exp((2u - u^2)X_0 + u^2X_1 + 1/2u^2[X_0, X_1]), \tag{2}$$

with $\alpha(0) = g_0$, $\alpha(1) \approx g_2$.

The value of $\alpha(1)$ is obtained from equation (2) as follows:

$$\begin{aligned}\alpha(1) &\approx g_0 \exp (X_0 + X_1 + 1/2\ [X_0, X_1]) \\ &\approx g_0 \exp X_0 \exp X_1 \text{ [from equation (1)]} \\ &= g_0(g_0^{-1}g_1)(g_1^{-1}g_2) \\ &= g_2.\end{aligned}$$

We turn now to the derivation of equation (2),

$$\begin{aligned}\alpha(u) &= g_0^2(u) = g_0 \exp uX_0 \exp uX_0^1 \\ &\approx g_0 \exp\left(uX_0 + uX_0^1 + \frac{u^2}{2}[X_0, X_0^1]\right)\end{aligned}$$

Now $x_0^1(u) = \log[(g_0^1)^{-1}g_1^1]$

and

$$\begin{aligned}(g_0^1)^{-1}g_1^1 &= (\exp uX_0)^{-1}g_0^{-1}g_1 \exp uX_1 \\ &= (\exp uX_0)^{-1} \exp X_0 \exp uX_1 \\ &= \exp(1 - u)X_0 \exp uX_1 \\ &\approx \exp((1 - u)X_0 + uX_1 + 1/2u(1 - u)[X_0, X_1]).\end{aligned}$$

So

$$X_0^1(u) = \log[(g_0^1)^{-1}g_1^1]$$

and

$$\approx (1 - u)X_0 + uX_1 + 1/2\ u(1 - u)[X_0, X_1]$$

and

$$[X_0, X_0^1] \approx u[X_0, X_1] + 1/2u(1 - u)[X_0,[X_0, X_1]].$$

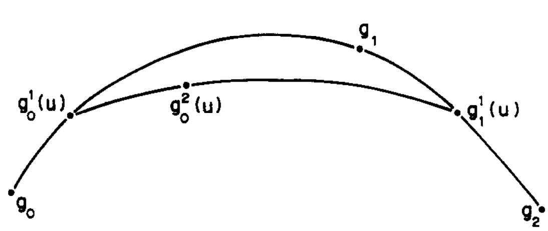

Fig. 7

We have then

$$\begin{aligned}\alpha(\mathrm{u}) &\approx g_0 \exp(\mathrm{u}X_0 + \mathrm{u}(1-\mathrm{u})X_0 + \mathrm{u}^2X_1 + 1/2^2(1-\mathrm{u})[X_0, X_1] \\ &\quad + \frac{\mathrm{u}^3}{2}[X_0, X_1] + \frac{\mathrm{u}^3}{4}(1-\mathrm{u})[X_0, [X_0, X_1]] \\ &\approx g_0 \exp((2\mathrm{u}-\mathrm{u}^2)X_0 + \mathrm{u}^2X_1 + 1/2\,\mathrm{u}^2[X_0, X_1])\end{aligned}$$

by ignoring third order commutators.

Remarks. We consider the special case when g_0, g_1, g_2 commute. This is satisfied, for example when these elements lie on a common one-parameter subgroup. In this case $[X_0, X_1] = 0$ and we have

$$\begin{aligned}\alpha(\mathrm{u}) &\approx g_0\exp((2\mathrm{u}-\mathrm{u}^2)X_0 + \mathrm{u}^2X_1) \\ &= g_0 \exp(2\mathrm{u}-\mathrm{u}^2)X_0 \exp {}^2X_1 \\ &= g_0(\exp X_0)^{2\mathrm{u}-\mathrm{u}^2}(\exp X_1)^{(\mathrm{u}^2)} \\ &= g_0(g_0^{-1}\, g_1)^{2\mathrm{u}-\mathrm{u}^2}(g_1^{-1}g_2)^{\mathrm{u}^2} \\ &= g_0^{1-2\mathrm{u}+\mathrm{u}^2} g_1^{2\mathrm{u}(1-\mathrm{u})} g_2^{\mathrm{u}^2} \\ &= g_0^{(1-\mathrm{u})^2} g_1^{2\mathrm{u}(1-\mathrm{u})} g_2^{\mathrm{u}^2}\end{aligned}$$

where the exponents are the Bernstein polynomials.

Next we wish to find the tangents to $\alpha(0)$ and $\alpha(1)$. We need one further formula from the theory of Lie algebras.

We have seen that

$$\frac{\mathrm{d}}{\mathrm{du}} \exp \mathrm{u}A = A \exp \mathrm{u}A,$$

but this formula is only valid for constant A.

If $A(\mathrm{u})$ is a smooth function of u, we have the general result

$$\frac{\mathrm{d}}{\mathrm{du}} \exp A(\mathrm{u}) = \exp A(\mathrm{u})\left(\frac{1-\exp(-\mathrm{ad}\, A(\mathrm{u}))}{\mathrm{ad}\, A(\mathrm{u})}\right)A'(\mathrm{u}),$$

where $\mathrm{ad}\, A(B) \equiv [A, B]$ and $A'(\mathrm{u}) = \mathrm{d}/\mathrm{du}\, A(\mathrm{u})$, so

$$\frac{\mathrm{d}}{\mathrm{du}} \exp A(\mathrm{u}) \approx \exp A(\mathrm{u})\left(A'(\mathrm{u}) - \frac{[A(\mathrm{u}), A'(\mathrm{u})]}{2}\right).$$

Since $\alpha(\mathrm{u}) \approx g_0 \exp((2\mathrm{u}-\mathrm{u}^2)X_0 + \mathrm{u}^2X_1 + 1/2\,\mathrm{u}^2[X_0, X_1])$ we let $A(\mathrm{u}) = (2\mathrm{u}-\mathrm{u}^2)X_0 + \mathrm{u}^2X_1 + 1/2\,\mathrm{u}^2[X_0, X_1]$. Then $A'(\mathrm{u}) = (2-2\mathrm{u})X_0 + 2\mathrm{u}X_1 + \mathrm{u}[X_0, X_1]$ and $[A(\mathrm{u}), A'(\mathrm{u})] \approx 2\mathrm{u}^2[X_0, X_1]$, so

$$\frac{\mathrm{d}}{\mathrm{du}} \exp A\mathrm{u} \approx \exp A(\mathrm{u})((2-2\mathrm{u})X_0 + 2\mathrm{u}X_1 + \mathrm{u}[X_0, X_1] - \mathrm{u}^2[X_0, X_1]).$$

Hence

$$\alpha'(\mathrm{u}) \approx g_0 \exp((2\mathrm{u}-\mathrm{u}^2)X_0 + \mathrm{u}^2X_1 + 1/2\mathrm{u}^2[X_0, X_1]) \times ((2-2\mathrm{u})X_0 + 2\mathrm{u}X_1 + (\mathrm{u}-\mathrm{u}^2)[X_0, X_1])$$

and

$$\begin{aligned}\alpha'(0) &= 2g_0X_0, \\ \alpha'(1) &\approx 2g_2X_1.\end{aligned}$$

So the tangents to $\alpha(\mathrm{u})$ at the endpoints are parallel to the tangents to the one-parameter subgroups $\beta_1(\mathrm{u}) = g_0 \exp \mathrm{u}X_0$ and $\beta_2(\mathrm{u}) = g_2 \exp \mathrm{u}X_1$, but have twice the length. Thus $\alpha(\mathrm{u})$ passes through its endpoints at twice the speed of the corresponding one-parameter subgroups, a result which is also true of Bezier curves.

4.3. *Four point interpolation*

In order to construct splines on GL_n, we need to use a four point Bezier construction. Let g_0, g_1, g_2, g_3 in GL_n be given and define

$$A_0 = g_0^{-1}g_1, \quad X_0 = \log A_0 = \log g_0^{-1}g_1,$$
$$B_0 = g_1^{-1}g_2, \quad Y_0 = \log B_0 = \log g_1^{-1}g_2,$$
$$C_0 = g_2^{-1}g_3, \quad Z_0 = \log C_0 = \log g_2^{-1}g_3,$$

where the above logarithms are defined for $g_0, \dots, g_3$ sufficiently close.

Next set, for $0 \leqslant u \leqslant 1$,

$$g_0^1(u) = g_0 \exp X_0,$$
$$g_1^1(u) = g_1 \exp Y_0,$$
$$g_2^1(u) = g_2 \exp Z_0,$$

the one-parameter paths connecting g_0, g_1 and g_1, g_2 and g_2, g_3.

The second Bezier iteraion proceeds by setting

$$A_1 = (g_0^1(u))^{-1}g_1^1(u), \quad X_1 = \log A_1 = \log[(g_0^1)^{-1}g_1^1],$$
$$B_1 = (g_1^1(u))^{-1}g_2^1(u), \quad Y_1 = \log B_1 = \log[(g_1^1)^{-1}g_2^1],$$

and defining

$$g_0^2(u) = g_0^1(u) \exp uX_1,$$
$$g_1^2(u) = g_1^1(u) \exp uY_1,$$

representing one-parameter curves connecting g_0^1 to g_1^1 and g_1^1 to g_2^1.

The final iteration is now obtained by setting

$$A_2 = (g_0^2(u))^{-1}g_1^2(u), \quad X_2 = \log A_2 = \log[(g_0^2)^{-1}g_1^2]$$

and defining

$$\alpha(u) \equiv g_0^3(u) = g_0^2(u) \exp X_2$$

(see Fig. 8).

We turn now to the approximation of $\alpha(u)$ in terms of the given values $g_0, \dots, g_3$:

$$\begin{aligned}
\alpha(u) &= g_0^3(u) = g_0^2(u) \exp uX_2 \\
&= g_0^1(u) \exp uX_1 \exp uX_2 \\
&= g_0 \exp uX_0 \exp uX_1 \exp uX_2 \\
&\approx g_0 \exp uX_0 \exp\left(uX_1 + uX_2 + \frac{u^2}{2}[X_1, X_2]\right) \\
&\approx g_0 \exp\left(uX_0 + uX_1 + uX_2 + \frac{u^2}{2}[X_0, X_1] + \frac{u^2}{2}[X_1, X_2] + \frac{u_2}{2}[X_0, X_2]\right).
\end{aligned}$$

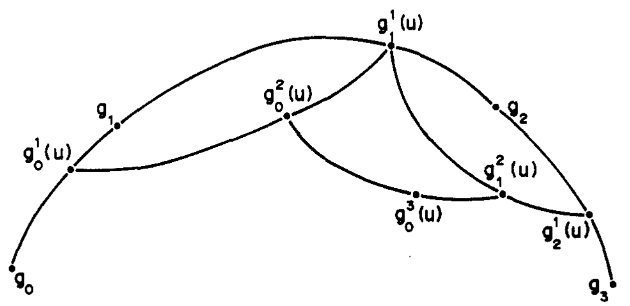

Fig. 8

Set

$$Z(\mathrm{u}) = \mathrm{u}X_0 + \mathrm{u}X_1 + \mathrm{u}X_2 + \frac{\mathrm{u}^2}{2}[X_0, X_1] + \frac{\mathrm{u}^2}{2}[X_1, X_2] + \frac{\mathrm{u}^2}{2}[X_0, X_2].$$

Now, by a calculation similar to that in Section 4.2, we find

$$X_1 \approx (1-\mathrm{u})X_0 + \mathrm{u}Y_0 + 1/2\mathrm{u}(1-\mathrm{u})[X_0, Y_0] \tag{3}$$

$$X_2 \approx (1-\mathrm{u})X_1 + \mathrm{u}Y_1 + 1/2\mathrm{u}(1-\mathrm{u})[X_1, Y_1] \tag{4}$$

and

$$Y_1 \approx (1-\mathrm{u})Y_0 + \mathrm{u}Z_0 + 1/2\mathrm{u}(1-\mathrm{u})[Y_0, Z_0]. \tag{5}$$

Then from equations (3) and (5)

$$\begin{aligned} X_2 \approx{} & (1-\mathrm{u})^2X_0 + \mathrm{u}(1-\mathrm{u})Y_0 + 1/2\mathrm{u}(1-\mathrm{u})^2[X_0, Y_0] \\ & + \mathrm{u}(1-\mathrm{u})Y_0 + \mathrm{u}^2Z_0 + 1/2\mathrm{u}^2(1-\mathrm{u})[Y_0, Z_0] \\ & + 1/2\mathrm{u}(1-\mathrm{u})[X_1, Y_1]. \end{aligned} \tag{6}$$

But, again using (3) and (5),

$$[X_1, Y_1] \approx (1-\mathrm{u})^2[X_0, Y_0] + \mathrm{u}(1-\mathrm{u})[X_0, Z_0] + \mathrm{u}^2[Y_0, Z_0] \tag{7}$$

up to quadratic commutators. So, from (6) and (7),

$$\begin{aligned} X_2 ={} & (1-\mathrm{u})^2X_0 + 2\mathrm{u}(1-\mathrm{u})Y_0 + \mathrm{u}^2Z_0 \\ & + 1/2\mathrm{u}(1-\mathrm{u})^2(2-\mathrm{u})[X_0, Y_0] \\ & + 1/2\mathrm{u}^2(1-\mathrm{u}^2)[Y_0, Z_0] \\ & + 1/2\mathrm{u}^2(1-\mathrm{u})^2[X_0, Z_0]. \end{aligned}$$

Finally, we need the commutators

$$\begin{aligned} & [X_0, X_1] \approx \mathrm{u}[X_0, Y_0], \\ & [X_0, X_2] \approx 2\mathrm{u}(1-\mathrm{u})[X_0, Y_0] + \mathrm{u}^2[X_0, Z_0], \\ & [X_1, X_2] \approx \mathrm{u}(1-\mathrm{u})^2[X_0, Y_0] + \mathrm{u}^3[Y_0, Z_0] \\ & \qquad + \mathrm{u}^2(1-\mathrm{u})[X_0, Z_0]. \end{aligned}$$

Substituting the above expression into $\alpha(\mathrm{u}) = g_0 \exp Z(\mathrm{u})$, we find

$$Z(\mathrm{u}) = \mathrm{u}(3-3\mathrm{u}+\mathrm{u}^2)X_0 + \mathrm{u}^2(3-2\mathrm{u})Y_0 + \mathrm{u}^3Z_0 + \frac{\mathrm{u}^2}{2}(3-2\mathrm{u})[X_0, Y_0] + \frac{\mathrm{u}^3}{2}[Y_0, Z_0] + \frac{\mathrm{u}^3}{2}[X_0, Z_0].$$

.

Thus we obtain

$$\begin{aligned} \alpha(\mathrm{u}) = g_0 \exp(& \mathrm{u}(3-3\mathrm{u}+\mathrm{u}^2)X_0 + \mathrm{u}^2(3-2\mathrm{u})Y_0 + \mathrm{u}^3Z_0 \\ & + \frac{\mathrm{u}^2}{2}(3-2\mathrm{u})[X_0, Y_0] + \frac{\mathrm{u}^3}{2}[Y_0, Z_0] + \frac{\mathrm{u}^3}{2}[X_0, Z_0], \end{aligned} \tag{8}$$

where $\alpha(0) = g_0$ and $\alpha(1) \approx g_3$, so that $\alpha(\mathrm{u})$ passes through the endpoints g_0 and g_3.

Further, a calculation similar to that in equation (6) shows that

$$\frac{\mathrm{d}}{\mathrm{du}}\alpha(\mathrm{u})\,\Big|_{\mathrm{u}=0} = 3g_0X_0,$$

$$\frac{\mathrm{d}}{\mathrm{du}}\alpha(\mathrm{u})\,\Big|_{\mathrm{u}=1} = 3g_3Z_0,$$

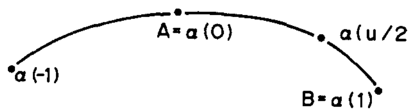

Fig. 9

so that at the endpoints $\alpha(u)$ is tangent to the one-parameter curves

$$\beta_1(u) = g_0 \exp uX_0$$

and

$$\beta_2(u) = g_3 \exp uZ_0$$

but passes through these points with three times the velocity, a result also noted in [5] for the case of spherical interpolation.

Finally, we note that when $g_0, \ldots, g_3$ commute, the expression for $\alpha(u)$ reduces to

$$\alpha(u) = g_0^{(1-u)^3} g_1^{3u(1-u)^2} g_2^{3u^2(1-u)} g_3^{u^3},$$

where the exponents are given by Bezier polynomials of third order.

5. CONSTRUCTION OF BEZIER SPLINES

Equation (8) of the previous section provides a Bezier interpolation of four points by a smooth curve lying in a given transformation group and passing through the two endpoints. The directions of this curve at its ends are determined by the tangents to one parameter subgroups through the end and middle points. In order to construct Bezier splines, we need to describe how to add intermediate points a_k, b_k to a given set of transformations $g_0, \ldots, g_n$. We can again imitate the construction of Section 2, using one-parameter subgroups in place of parametrized straight lines.

If $\alpha(u)$ is a one-parameter path joining A to B, then $\alpha(u/2)$ represents the "midpoint" of this path, and $\alpha(-1)$ the "reflection" of B through A (Fig. 9). Thus for given transformations g_{k-1}, g_k, g_{k+1}, let $A = \log g_k^{-1} g_{k-1}$ and $\alpha(u) = g_k \exp uA$ be the one-parameter curve joining g_k and g_{k-1}. Then $B = \alpha(-1)$ is the reflection of g_{k-1} through g_k. Next set $C = \log B^{-1} g_{k+1}$ and define $\beta(u) = B \exp uC$, the one-parameter curve joining $\alpha(-1)$ and g_{k+1}.

The first intermediate point a_k is given by

$$a_k = \beta(u/2).$$

Setting $D = \log a_k^{-1} g_k$, $\gamma(u) = a_k \exp uD$ is the one-parameter curve joining a_k and g_k, and its reflection defines the second intermediate point

$$b_k = \gamma(-1).$$

Now for each quadruple g_k, a_k, b_{k+1}, g_{k+1} we can construct a Bezier curve joining g_k and g_{k+1} using the four point interpolation formula. These curve segments, parametrized for $0 \leqslant u \leqslant 1$, will join smoothly together to give a Bezier spline connecting the original transformations $g_0, \ldots, g_n$.

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